

# Modules over Strongly Semiprime Rings

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**Abstract.** For a ring  $A$ , the following conditions are equivalent.

- 1)  $A$  is a right strongly semiprime ring.
- 2) Every right  $A$ -module which is injective with respect to some essential right ideal of the ring  $A$ , is an injective module;
- 3) Every quasi-injective right  $A$ -module which is injective with respect to some essential right ideal of the ring  $A$  is an injective module.

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**Key words:** injective module; strongly semiprime ring; quasi-injective module

## 1. Introduction and preliminaries

All rings are assumed to be associative and with zero identity element; all modules are unitary and, unless otherwise specified, all modules are right modules. This paper is a continuation of [7].

For a module  $Y$ , a module  $X$  is said to be *injective with respect to  $Y$*  or  *$Y$ -injective* if for each submodule  $Y_1$  of  $Y$ , every homomorphism  $Y_1 \rightarrow X$  can be extended to a homomorphism  $Y \rightarrow X$ . A module is said to be *injective* if it is injective with respect to each module. A module  $X$  is said to be *quasi-injective* if  $X$  is injective with respect to  $X$ . Every finite cyclic group is a quasi-injective non-injective module over the ring of integers  $\mathbb{Z}$ .

**Remark 1.1.** The following *Baer's criterion*<sup>2</sup> is well-known: if  $A$  is a ring and  $X$  is a right  $A$ -module, then  $X$  is injective if and only if  $X$  is injective with respect to the module  $A_A$ .

A ring  $A$  is said to be *right strongly semiprime* [2] if each ideal of  $A$  which is an essential right ideal contains a finite subset with zero right annihilator. A ring  $A$  is said to be *right strongly prime* [3] if every non-zero ideal of  $A$  contains a finite subset with zero right annihilator. It is clear that every right strongly prime ring is right strongly semiprime. The direct product of two finite fields is a finite commutative strongly semiprime ring which is not strongly prime.

**Remark 1.2.** Let  $A$  be a right strongly prime ring and let  $X$  be a right  $A$ -module. In [7], it is proved that  $X$  is injective if and only if  $X$  is injective with respect to some non-zero right ideal of the ring  $A$ .

For a module  $X$ , a submodule  $Y$  of  $X$  is said to be *essential* in  $X$  if  $Y \cap Z \neq 0$  for each non-zero submodule  $Z$  of  $X$ . A right module  $X$  over the ring  $A$  is said to be *non-singular* if the right annihilator  $r(x)$  of any non-zero element  $x \in X$  is not essential right ideal of the ring  $A$ . For a module  $X$ , we denote by  $G(X)$  or  $\text{Sing}_2 X$  the intersection of all submodules  $Y$  of the module  $X$  such that the factor module  $X/Y$  is non-singular. The submodule  $G(X)$  is a fully

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<sup>2</sup>For example, see [6, Proposition 10.3'].

invariant submodule of  $X$  and is called the *Goldie radical* or the *second singular submodule* of the module  $X$ .

In connection to Remark 1.1 and Remark 1.2, we prove Theorem 1.3 and Theorem 1.4 which are the main results of this paper.

**Theorem 1.3.** *For a given ring  $A$  with right Goldie radical  $G(A_A)$ , the following conditions are equivalent.*

- 1) *Every non-singular right  $A$ -module  $X$  which is injective with respect to some essential right ideal of the ring  $A$  is an injective module.*
- 2)  *$A/G(A_A)$  is a right strongly semiprime ring.*

**Theorem 1.4.** *For a given ring  $A$ , the following conditions are equivalent.*

- 1)  *$A$  is a right strongly semiprime ring;*
- 2) *Every right  $A$ -module which is injective with respect to some essential right ideal of the ring  $A$ , is an injective module and  $A$  is right non-singular.*

**Remark 1.5.** In connection to Theorem 1.3 and Theorem 1.4, we note that there exist a finite commutative ring  $A$ , an essential ideal  $B$  of the ring  $A$ , and a non-injective  $B$ -injective  $A$ -module  $X$ . We denote by  $A$ ,  $B$  and  $X$  the finite commutative ring  $\mathbb{Z}/4\mathbb{Z}$ , the ideal  $2\mathbb{Z}/4\mathbb{Z}$  and the module  $B_A$ , respectively. Then  $B$  is an essential ideal and the module  $X$  is injective with respect to  $B_A$ . Since  $X$  is not a direct summand of  $A_A$ , the module  $X$  is not injective.

**Remark 1.6.** A ring without non-zero nilpotent ideals is called a *semiprime* ring. Every right strongly semiprime ring is a right non-singular semiprime ring [2]. The direct product of infinitely many fields is an example of a commutative semiprime non-singular ring which is not strongly semiprime. All finite direct products of rings without zero-divisors and all finite direct products of simple rings are right and left strongly semiprime rings.

**Remark 1.7.** A ring  $A$  is called a *right Goldie ring* if  $A$  is a ring with the maximum condition on right annihilators which does not contain the direct sum of infinitely many non-zero right ideals. If  $A$  is a semiprime right Goldie ring, then it is well known <sup>3</sup> that every essential right ideal of the ring  $A$  contains a non-zero-divisor. Therefore, all semiprime right Goldie rings are right strongly semiprime. In particular, all right Noetherian semiprime rings are right strongly semiprime.

We denote by  $\text{Sing } X$  the *singular submodule* of the right  $A$ -module  $X$ , that is  $\text{Sing } X$  is the fully invariant submodule of  $X$  consisting of all elements  $x \in X$  such that  $r(x)$  is an essential right ideal of the ring  $A$ . A module  $X$  is said to be *singular* if  $X = \text{Sing } X$ . A module  $X$  is called a *Goldie-radical* module if  $X = G(X)$ . The relation  $G(X) = 0$  is equivalent to the property that the module  $M$  is non-singular. We use well known properties of  $\text{Sing } X$  and  $G(X)$ ; e.g., see [1]. A submodule  $Y$  of the module  $X$  is said to be *closed* in  $X$  if  $Y = Y'$  for each submodule  $Y'$  of  $X$  which is an essential extension of the module  $Y$ .

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<sup>3</sup>For example, see [6, Theorem 3.2.14].

## 2. The proof of Theorem 1.3 and Theorem 1.4

The proof of Theorem 1.3 and Theorem 1.4 is decomposed into a series of assertions, some of which are of independent interest.

**Lemma 2.1.** *Let  $A$  be a ring and let  $X$  be a right  $A$ -module.*

1. *If  $B$  is a right ideal of the ring  $A$  and the module  $X$  is injective with respect to the module  $B_A$ , then  $X$  is injective with respect to the module  $(AB)_A$ , where  $AB$  is the ideal generated by the right ideal  $B$ . In addition, if the ideal  $AB$  contains a finite subset  $C$  with  $r(C) = 0$ , then the module  $X$  is injective.*
2. *If  $X \neq G(X)$ , then there exists non-zero right ideal  $B$  of the ring  $A$  such that the module  $B_A$  is isomorphic to a submodule of the module  $X$ .*
3. *Let  $Y$  be a right  $A$ -module,  $\{Y_i\}_{i \in I}$  be some set of right modules such that the module  $X$  is injective with respect to  $Y_i$  for each  $i \in I$  and let  $\{f_i \in \text{Hom}(Y_i, Y)\}_{i \in I}$  be some set of homomorphisms. Then the module  $X$  is injective with respect to the submodule  $\sum_{i \in I} f_i(Y_i)$  of the module  $Y$ . In addition, if there exists a monomorphism  $A_A \rightarrow \sum_{i \in I} f_i(Y_i)$ , then the module  $X$  is injective.*
4. *Let  $Y$  be a right  $A$ -module and let  $\{Y_i\}_{i \in I}$  be some set of submodules of the module  $Y$  such that the module  $X$  is injective with respect to  $Y_i$  for each  $i \in I$ . Then the module  $X$  is injective with respect to the submodule  $\sum_{i \in I} Y_i$  of the module  $Y$ . In addition, if there exists a monomorphism  $A_A \rightarrow \sum_{i \in I} Y_i$ , then the module  $X$  is injective.*

**Proof.** 1, 2, 3. The assertions are proved in Lemma 3, Lemma 4 and Lemma 2.2 from [7], respectively.

4. The assertion follows from 3 if we denote by  $f_i$  the natural embeddings  $Y_i \rightarrow Y$ .  $\square$

**Lemma 2.2.** *Let  $A$  be a ring,  $X$  be a non-singular non-zero right  $A$ -module,  $\{C_i \mid i \in I\}$  be the set of all non-zero right ideals of the ring  $A$  such that every non-zero submodule of the  $A$ -module  $C_i$  is not isomorphic to a submodule of the module  $X$ , and let  $\{D_j \mid j \in J\}$  be the set of all non-zero right ideals  $D_j$  of the ring  $A$  such that  $D_j$  is isomorphic to a submodule of the module  $X$ . We set  $C = \sum_{i \in I} C_i$ ,  $D = \sum_{j \in J} D_j$ , and  $B = C + D$ .*

1. *For any submodule  $C'$  of the module  $C_A$ , every homomorphism  $f: C'_A \rightarrow X$  is the zero homomorphism.*
2. *The module  $X$  is injective with respect to the module  $C_A$ .*
3.  *$B$  is an essential right ideal of the ring  $A$ .*
4. *If the module  $X$  is quasi-injective, then  $X$  is injective with respect to the essential right ideal  $B$ .*

**Proof.** 1. Let us assume that  $f \neq 0$ . Since  $X$  is a non-singular module and  $C'/\text{Ker } f \cong f(C') \subseteq X$ , we have that  $\text{Ker } f$  is not an essential submodule of

$C'_A$ . There exists a non-zero element  $c \in C'$  with  $cA \cap \text{Ker } f = 0$ . Then the non-zero submodule  $cA$  of the module  $C'$  is isomorphic to the non-zero submodule  $f(cA)$  of the module  $X$ . Therefore,  $f(c) \neq 0$ . There exists a finite subset  $K$  in  $I$  such that  $c = \sum_{k \in K} c_k$  and  $c_k \in C_k$  for all  $k \in K$ . Since  $f(c) \neq 0$ , we have that  $f(c_k) \neq 0$  for some  $k \in K \subseteq I$ . Therefore,  $c_k A$  is a non-zero submodule of the  $A$ -module  $C_k$  which is isomorphic to a non-zero submodule of the module  $X$ . This contradicts to the property that  $C_k \in \{C_i \mid i \in I\}$ .

2. The assertion follows from 1.

3. Let us assume that  $B$  is not an essential right ideal. Then  $B \cap E = 0$  for some non-zero right ideal  $E$ . Then  $C \cap E = 0$  and  $D \cap E = 0$ . Since  $C \cap E = 0$ , we have that  $E \notin \{C_i \mid i \in I\}$ . Therefore, there exists a non-zero submodule  $E_1$  of the module  $E$  which is isomorphic to a submodule of the module  $X$ . Then  $E_1 \in \{D_j \mid j \in J\}$ . Therefore,  $E_1 \subseteq D \cap E = 0$ . This is a contradiction.

4. Since  $X$  is a quasi-injective module,  $X$  is injective with respect to any module which is isomorphic to a submodule of the module  $X$ . Therefore,  $X$  is injective with respect to each of the  $A$ -module  $D_j$ . By Lemma 2.1(4), the module  $X$  is injective with respect to the module  $D_A$ . In addition,  $X$  is injective with respect to the module  $C_A$  by 2. By Lemma 2.1(4), the module  $X$  is injective with respect to the module  $C + D = B$ .  $\square$

**Proposition 2.3.** *Let  $A$  be a right strongly semiprime ring and  $X$  be a right  $A$ -module. If there exists an essential right ideal  $B$  of the ring  $A$  such that  $X$  is injective with respect to the module  $B_A$ , then  $X$  is an injective module.*

**Proof.** By Lemma 2.1(1),  $X$  is injective with respect to the module  $(AB)_A$ , where  $AB$  is the ideal generated by the right ideal  $B$ . Since  $B$  is an essential right ideal and  $B \subseteq AB$ , the ideal  $AB$  is an essential right ideal. Since  $A$  is a right strongly semiprime ring, the ideal  $AB$  contains a finite subset  $K = \{k_1, \dots, k_n\}$  with zero right annihilator  $r(K)$ . Since  $r(K) = r(k_1) \cap \dots \cap r(k_n) = 0$ , the module  $A_A$  is isomorphic to a submodule of the direct sum of  $n$  copies of the module  $(AB)_A$ . In addition, module  $X$  is injective with respect to the module  $(AB)_A$ . By Lemma 2.1(3), the module  $X$  is injective.  $\square$

For completeness, we briefly prove the following familiar lemma.

1. *If  $B$  is an essential right ideal of the ring  $A$ , then  $h(B)$  is an essential right ideal of the ring  $h(A)$ .*
2. *If  $B$  is a right ideal of the ring  $A$  such that  $G \subseteq B$  and  $h(B)$  is an essential right ideal of the ring  $h(A)$ , then  $B$  is an essential right ideal of the ring  $A$ .*
3.  *$MG \subseteq G(M)$  for each right  $A$ -module  $M$ .*
4.  *$XG = 0$  and a natural  $h(A)$ -module  $X$  is non-singular. In addition, if  $Y$  be an arbitrary non-singular right  $A$ -module, then  $YG = 0$  and the  $h(A)$ -module homomorphisms  $Y \rightarrow X$  coincide with the  $A$ -module homomorphisms  $Y \rightarrow X$ . Therefore,  $X$  is an  $Y$ -injective  $A$ -module if and only if  $X$  is an  $Y$ -injective  $h(A)$ -module.*
5.  *$X$  is an injective  $h(A)$ -module if and only if  $X$  is an injective  $A$ -module.*

6.  $X_{h(A)}$  is an essential extension of a direct sum of uniform modules if and only if  $X_A$  is an essential extension of a direct sum of uniform modules.

**Proof. 1.** Let us assume that  $h(B)$  is not an essential right ideal of the ring  $h(A)$ . Then there exists a right ideal  $C$  of the ring  $A$  such that  $C$  properly contains  $G$  and  $h(B) \cap h(C) = h(0)$ . Since  $h(B) \cap h(C) = h(0)$ , we have  $B \cap C \subseteq G$ . Since  $C$  properly contains the closed right ideal  $G$ , then  $C_A$  contains a non-zero submodule  $D$  with  $D \cap G = 0$ . Since  $B$  is an essential right ideal,  $B \cap D \neq 0$ ; in addition,  $(B \cap D) \cap G = 0$ . Then  $h(0) \neq h(B \cap D) \subseteq h(B) \cap h(C) = h(0)$ . This is a contradiction.

**2.** Let us assume that  $B$  is not an essential right ideal of the ring  $A$ . Then  $B \cap C = 0$  for some non-zero right ideal  $C$  of the ring  $A$  and  $G \cap C \subseteq B \cap C = 0$ . Therefore,  $h(C) \neq h(0)$ . Since  $h(B)$  is an essential right ideal of the ring  $h(A)$ , we have  $h(B) \cap h(C) \neq h(0)$ . Let  $h(0) \neq h(b) = h(c) \in h(B) \cap h(C)$ , where  $b \in B$  and  $c \in C$ . Then  $c - b \in G \subseteq B$ . Therefore,  $c \in B \cap C = 0$ , whence we have  $h(c) = h(0)$ . This is a contradiction.

**3.** For any element  $m \in M$ , the module  $mG_A$  is a Goldie-radical module, since  $mG_A$  is a homomorphic image of the Goldie radical module  $G$ . Therefore,  $mG \subseteq G(M)$  and  $MG \subseteq G(M)$ .

**4.** By 3,  $XG = 0$ . Let us assume that  $x \in X$  and  $xh(B) = 0$  for some essential right ideal  $h(B)$ , where  $B = h^{-1}(h(B))$  is the complete pre-image of  $h(B)$  in the ring  $A$ . By 2,)  $B$  is an essential right ideal of the ring  $A$ . Then  $xB = 0$  and  $x \in \text{Sing } X = 0$ . Therefore,  $X$  is a non-singular  $h(A)$ -module. The remaining part of 4 is directly verified.

**5.** Let  $R$  be one of the rings  $A$ ,  $h(A)$  and let  $M$  be a right  $R$ -module. By Lemma 2.1(4), the module  $M$  is injective if and only if  $M$  is injective with respect to the module  $R_R$ . Now the assertion follows from 4.

**6.** The assertion follows from 4. □

**Proposition 2.5.** *Let  $A$  be a ring and let  $G = A/G(A_A)$ . The following conditions are equivalent.*

- 1) *Every non-singular right  $A$ -module  $X$  which is injective with respect to some essential right ideal of the ring  $A$  is an injective module.*
- 2) *Every quasi-injective non-singular right  $A$ -module  $X$  which is injective with respect to some essential right ideal of the ring  $A$  is an injective module.*
- 3) *Every quasi-injective non-singular right  $A$ -module is an injective module.*
- 4)  *$A/G(A_A)$  is a right strongly semiprime ring.*

**Proof.** The implication  $1) \Rightarrow 2)$  is obvious.

The implication  $2) \Rightarrow 3)$  follows from Lemma 2.2(4).

The equivalence of 3) and 4) is proved in [4].

$4) \Rightarrow 1)$ . Let  $R$  be one of the rings  $A$ ,  $A/G(A_A)$  and let  $M$  be a right  $R$ -module. By Lemma 2.1(4), the module  $M$  is injective if and only if  $M$  is injective with respect to the module  $R_R$ .

Let  $h: A \rightarrow A/G$  be a natural ring epimorphism and let  $X$  be a non-singular right  $A$ -module which is injective with respect to some essential right ideal  $B$  of the ring  $A$ . By Lemma 2.4(4),  $XG = 0$  and  $X$  is a natural non-singular  $h(A)$ -module. By Lemma 2.4(1),  $h(B)$  is an essential ideal of the ring  $h(A)$ . By Lemma 2.4(4), the module  $X$  is injective with respect to  $h(B)$ . By Proposition 2.3,  $X$  is an injective  $h(A)$ -module. By Lemma 2.4(5),  $X$  is an injective  $A$ -module.  $\square$

**Remark 2.6. The completion of the proof of Theorem 1.3 and Theorem 1.4.** Theorem 1.3 follows from Proposition 2.5. Theorem 1.4 follows from Proposition 2.3 and the property that every right strongly semiprime ring is right non-singular [2].

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